2.1 复数的定义、几何意义及应用

2.1.1 复数的定义、复共轭和基本运算

定义 2.1

通过引入虚数单位 (imaginary unit) i

$$i^2 = -1,$$

得到复数 z = x + iy。 实部 x = Re(z) 虚部 y = Im(z)

复共轭
$$z^*$$
(or \bar{z}) $\equiv x - iy$, $i^* = -i$, $zz^* = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$ 于是有

实部:
$$x = Re(z) = \frac{z + z^*}{2}$$
 虚部: $y = Im(z) = \frac{z - z^*}{2i}$ (2.1)

重要公式 2.1

实数运算法则依然适用, 只不过虚数单位 i 要满足

$$i^2 = -1, i^* = -i$$

2023 年高考题

 $z = \frac{-3+i}{2+i}$ 的共轭复数是()

- A. 2 + i
- B. 2 i
- C. -1 + i
- D. -1 i

已知 $z = \frac{1-i}{2+2i}$, 则 $z - \bar{z} =$

- A. −i
- B. i
- C. 0
- D. 1

若 i(1-z) = 1, 则 $z + \bar{z} =$

- A. -2
- B. -1
- C. 1
- D. 2

已知 z = 1 - 2i, 且 $z + a\bar{z} + b = 0$, 其中 a, b 为实数, 则

A. a = 1, b = -2

B.
$$a = -1, b = 2$$

C.
$$a = 1, b = 2$$

D.
$$a = -1, b = -2$$

例题 2.1

$$z_1 = 1 + 2i \quad z_2 = 3 + 2i$$

$$z_1 + z_2 = 4 + 4i \quad z_1 - z_2 = -2$$

$$z_1 \cdot z_2 = 3 + 4i^2 + 6i + 2i = 3 - 4 + 8i = -1 + 8i$$

$$z_1^* z_1 = (1 + 2i)(1 - 2i) = 1 - 4i^2 = 5$$

$$z_2^* z_2 = (3 + 2i)(3 - 2i) = 9 - 4i^2 = 13$$

$$\frac{z_1}{z_2} = \frac{z_1 \cdot z_2^*}{z_2 \cdot z_2^*} = \frac{(1 + 2i)(3 - 2i)}{13} = \frac{3 - 4i^2 + 6i - 2i}{13} = \frac{7 + 4i}{13}$$

$$(z_1 + z_2)^4 = ?$$

$$\overrightarrow{D} - : \quad z_1 + z_2 = 4 + 4i = 4(1 + i)$$

$$(z_1 + z_2)^2 = 16(1 + i^2 + 2i) = 32i$$

$$(z_1 + z_2)^4 = 32^2 i^2 = -32^2$$

$$\overrightarrow{D} \stackrel{\text{A.T.}}{} : z_1 + z_2 = 4\sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = 4\sqrt{2}e^{i\frac{\pi}{4}}$$

$$(z_1 + z_2)^4 = (4\sqrt{2})^4 e^{i\pi} = 32^2(-1) = -32^2$$

2.1.2 复数的几何意义

复数是一对实数通过虚数单位 i 组合在一起,正好对应用平面直角坐标系 OXY 上的点,或者向量。称该平面为复平面 C^1 。X 轴叫实轴,Y 轴叫虚轴。

定义 2.2

复数的几何解释——复数即平面上的点或者向量: 复数 z = x + iy 可以看成平面直角坐标系 OXY 上的点 (x,y) 或者等价的从原点到此点的向量。

图 2.3中显示了复平面和复平面上的复数。

既然复数和平面直角坐标系的向量——对应,那么复数的加减运算也满足向量法则。 平面直角坐标系下,很方便加减运算,可以用向量法则。

 $^{^1}$ 复平面 (complex plane):高斯的论文正式确定了复平面这一重要概念,有了这个概念后,复变函数论迅猛发展。

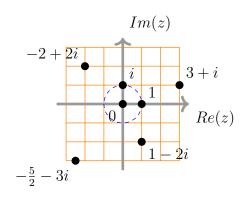


图 2.3: 复数和复平面上的点

例题 2.2

$$z_1 = 1 + 2i \quad z_2 = 3 + 2i$$

$$z_1 + z_2 = 4 + 4i$$

方法二: $z_1 + z_2 = 4\sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = 4\sqrt{2}e^{i\frac{\pi}{4}}$

$$(z_1 + z_2)^4 = (4\sqrt{2})^4 e^{i\pi} = 32^2(-1) = -32^2$$

例题 2.3 证明:

$$|z_1 + z_2| \leqslant |z_1| + |z_2|$$

利用复数的几何意义:三角形两边之和大于第三边。

重要公式 2.2

复数的平面极坐标表示:采用平面上的极坐标 r, θ , 类比向量, 定义复数的模和幅角。

$$z = r\cos\theta + i \ r\sin\theta = \frac{\text{wtick}}{\cos\theta + i\sin\theta = e^{i\theta}} \ re^{i\theta}$$

|z|=r 是模

 $arg(z) = \theta$ 是幅角

这样我们得到了复数的极坐标和复指数表示,这样的对于乘除运算和幂次运算有优势.

三角函数的周期性会导致幅角的多值性,即 θ 和 $\theta+2n\pi$ $(n=0,\pm 1,\pm 2,...)$ 对三角函数或者复指数一样。

注意我们本课用小写的 arg 表示辐角, 大写的 Arg 表示辐角主值。

$$\arg(z) = \operatorname{Arg}(z) + 2n\pi, \quad n = 0, \pm 1, \pm 2, \dots$$
 (2.2)

幂次运算:复指数形式用起来方便,也容易看出复数乘法的几何意义:模相乘,幅角相加。

但对分数幂次,会出现起源于幅角周期性的多值特征(更多的多值函数的讨论将放在一个月之后)

例题 2.4 计算 $(1+i\sqrt{3})^{1/2}$

解

$$1 + \sqrt{3}i = 2e^{i\left(\frac{\pi}{3} + 2k\pi\right)}$$

$$(1 + \sqrt{3}i)^{\frac{1}{2}} = \sqrt{2}e^{i\left(\frac{\pi}{6} + k\pi\right)} = (-1)^k\sqrt{2}e^{i\frac{\pi}{6}}$$
(2.3)

性质

$$e^{i\theta} \cdot e^{i\phi} = e^{i(\theta + \phi)} \tag{2.4}$$

$$\frac{d}{d\theta}e^{i\theta} = ie^{i\theta} \tag{2.5}$$

重要公式 2.3

简谐运动方程的解 $f(x)'' + k^2 f(x) = 0 \rightarrow f(x) = e^{ikx}$

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重要公式 2.4

$$e^{i\theta} = \cos\theta + i\sin\theta, e^{i\pi} + 1 = 0$$

欧拉公式被克莱因称之为数学中最卓越的公式2.

例题 2.5 结合下列几何图形加深复数几何意义的理解:

- $(1) |z| = 1 \Rightarrow 单位圆$
- (2) |z+3|+|z-3|=10 ⇒ 椭圆 $a=5, c=3, b=\sqrt{a^2-c^2}=4$ $\frac{x^2}{25}+\frac{y^2}{16}=1$ 例题 2.6 在复平面上画出 $z_1=\mathrm{e}^{i\pi/4}, z_2=z_1^2$ 及其和 z_1+z_2 ,并计算 $\tan\frac{3\pi}{8}$ 解 如图 2.4所示:

对于复指数形式的 z_1 的平方, $z_2 = e^{i\pi/4\times2} = e^{i\pi/2} = i$ 运用向量相加的平行四边形法则,可以容易计算出

$$z_1 + z_2 = \frac{\sqrt{2}}{2} + i(1 + \frac{\sqrt{2}}{2})$$

从图上很容易读出 $\tan \frac{3\pi}{8}$ 就是向量 $z_1 + z_2$ 的虚部除以实部

$$\tan\frac{3\pi}{8} = 1 + \sqrt{2}.$$

欧拉公式可以用来推导出

重要公式 2.5

第莫佛公式:

$$e^{iP\varphi} = (\cos\varphi + i\sin\varphi)^P = \cos(P\varphi) + i\sin(P\varphi).$$
 (2.6)

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$$e^{i\pi} + 1 = 0$$

这五个数是人类文明思想发展之中的伟大创见。

- 0:一切观测的立足点
- 1: 所有丈量测度的基准
- π: 解读周期現象与统计规律的因子
- i: 从虚无到实在的突破
- e: 常见的演化因子

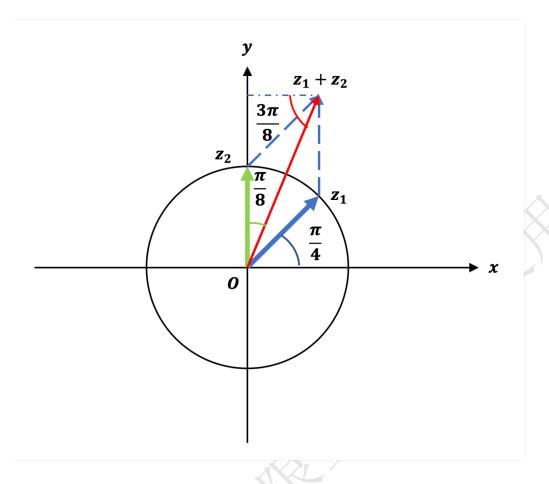


图 2.4: 用向量法则进行复数求和

这里一般 P 为整数,对于非整数,会出现多值特征。

欧拉公式和第莫佛公式是三角函数恒等式的强大生成器。见后面的例题和习题。

例题 2.7 求证

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \sin\phi\cos\theta$$

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
(2.7)

证明

$$e^{i\theta} \cdot e^{i\phi} = e^{i(\theta + \phi)} \tag{2.8}$$

$$(\cos \theta + i \sin \theta)(\cos \phi + i \sin \phi) = \cos(\theta + \phi) + i \sin(\theta + \phi)$$
 (2.9)

 $\cos\theta\cos\phi - \sin\theta\sin\phi + i(\sin\theta\cos\phi + i\sin\phi\cos\phi) = \cos(\theta + \phi) + i\sin(\theta + \phi)$ 左右两边实部与虚部分别相等得到

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \sin\phi\cos\theta$$

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
(2.10)

例题 2.8 求和

$$S = \sum_{n=1}^{M} \sin n\theta = \sin \theta + \sin 2\theta + \dots + \sin M\theta$$
 (2.11)

解 利用欧拉公式

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{2.12}$$

$$S = \operatorname{Im}(e^{i\theta}) + \operatorname{Im}(e^{i2\theta}) + \cdots \operatorname{Im}(e^{iM\theta})$$
$$= \operatorname{Im}\left(\sum_{n=1}^{M} e^{in\theta}\right) \equiv \operatorname{Im}P$$
 (2.13)

$$P = \sum_{n=1}^{M} e^{in\theta}$$
等比级数
$$\frac{e^{i\theta}}{e^{i\theta}} = e^{i\theta} \frac{e^{iM\theta} - 1}{e^{i\theta} - 1}$$
要理分母的常用技巧
$$\frac{e^{-i\theta/2}}{e^{-i\theta/2}} = \frac{e^{i\frac{\theta}{2}} \left(e^{iM\theta} - 1\right)}{e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}} = \frac{e^{i\left(\frac{2M+1}{2}\right)\theta} - e^{i\frac{\theta}{2}}}{2i\sin\frac{\theta}{2}}$$
=
$$\frac{i\left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2} - \cos\frac{2M+1}{2}\theta - i\sin\frac{2M+1}{2}\theta\right]}{2\sin\frac{\theta}{2}}$$
(2.14)

取虚部

$$S = \sum_{n=1}^{M} \sin n\theta = \text{Im}(P) = \frac{\cos \frac{\theta}{2} - \cos \frac{2M+1}{2}\theta}{2\sin \frac{\theta}{2}}$$
 (2.15)

买一送一:

取实部

$$\sum_{n=1}^{M} \cos n\theta = \operatorname{Re}(P) = \frac{-\sin\frac{\theta}{2} + \sin\frac{2M+1}{2}\theta}{2\sin\frac{\theta}{2}}$$
(2.16)

1. 证明

$$\cos 3\theta = 4C^3 - 3C, \quad \sin 3\theta = -4S^3 + 3S. \tag{2.17}$$

其中 $C \equiv \cos \theta$, $S \equiv \sin \theta$

2. 计算

$$\int_0^{\pi} \cos^n \theta \cos n\theta \ d \ \theta$$

3. 计算

ln i

4. 计算

 i^i

课程预告

- □ 解答一些上节课的问题
- □ 课前十分钟四位同学到黑板上不看 讲义计算,如果有同学觉得例题简单 的话,可以做一道上面的习题代替我 指定的例题-

汤新源: 计算 $(4+i4)^4$;

曾欣怡: 计算 $(1+i\sqrt{3})^{1/2}$;

常一界: 在复平面上画出 $e^{i\pi/4}, z_2 = z_1^2$ 及其和 $z_1 + z_2$, 并计算 $\tan \frac{3\pi}{8}$

陈启洋: 计算 $\sum_{n=1}^{M} \sin n\theta$;

□ 后面接着学习复变函数可导和解析 的条件(柯西-黎曼条件)及其最重要 的应用——得到复变函数的核心,即 柯西定理, 然后应用柯西定理在复数 域内求菲涅尔积分。对应教材上 2.1

2.2 2.3 3.1 3.2 节



思考题

- \diamondsuit 将 $\cos 5x$ 及 $\sin 5x$ 展开为 $\cos x$, $\sin x$ 的表达式,并计算 $\cos \frac{2\pi}{5}$ 及 $\sin \frac{2\pi}{5}$
- ♦ 解方程 $x^{n-1} + x^{n-2} + \cdots + x + 1 = 0$ (称为割圆方程) 并给出方程 $x^4 + x^3 + x^2 + x + 1 = 0$ 0 每一个根的值。

₩ 课外阅读 皿

参考陈先生微分几何讲义

第一讲微积分在复变函数论中应用简介

我还应该再讲两次. 这两次我有个计划: 预备讲一点复变函数论, 因为在数学中, 很要紧 的一件事实,同时在数学史上也是非常要紧的一件事情,就是有复数.这个复数使得数学简单, 复函数有许多漂亮, 有意思的性质, 因此, 这使得这些函数在应用上特别有用处. 所以, 我预 备讲一讲, 比如说, 复变函数有一个很重要的性质: 任意的代数方程在复变函数之中一定有解. 这是一个不得了的事情, 因为不管你怎么样写一个方程, 你要是允许解是复数的话, 它一定有 解. 例如, $x^2 + 1 = 0$, 那么它有个解就是 $\sqrt{-1}$, 所以 $\sqrt{-1}$ 就这么样子有用处. 不但如此, 复 数跟实数一样, 可以加减, 有同样的性质, 所以, 它可以运算. 同时它包含了许多材料是实数不 能包含的. 我想我的课在过程中一定会有个空挡, 在空挡的时候, 我想找两次讲复变函数. 我 预备讲:一个是我刚才讲的代数的基本定理,就是说任意的代数的方程在复数域中一定有解. 这个是很难证明的, 需要数学上新的观念. 比方说, 伟大数学家如 Euler, 他想法子证明, 但没 有能成功. 我想 Gauss 是我们近代最伟大的数学家, 他很年轻的时候就有一个证明, 也就是复 数需要一些几何的性质, 不完全是代数的问题. 我预备下次讲复数的时候证明这个定理; 同时, 复变函数最主要的一个定理是 Picard 定理, 就是说, 假使对于一个复变函数, 取它的函数值 在复平面里头所取的位置, 它把整个复平面都盖住了, 其中也许可以去掉一点, 两点. 这是不 得了的,就是说,函数如果是一个全纯函数的话,它分布得非常之均匀,可以说差不多把平面 都盖住了. 有意思的一件事情是这个定理是复变函数高峰的定理, 可以利用我们现在要讲的 Gauss-Bonnet 公式来证明. 这说明看起来没有关系的一些方法跟观念, 结果是有关系的. 这 是数学上非常要紧, 有意思的问题.

第五讲

"我想微积分有一个重要的应用是在复变函数论. 你要讲数, 最有意思的数就是复数. 很惭愧, 我看中国数学史, 中国人太实际了, 中国人没有复数. 复数要紧得不得了. 我要讲一点复变函数, 我要证代数基本定理. 复变函数之后, 任何代数方程式都有解, 这是不得了的一个结果. 这个结果, 当年 Euler 不会证, 很多人都证不出来. Gauss 能证明, 他是近代最伟大的数学家."

第二讲

微积分还有一个发展, 最要紧的是复数. 很奇怪的, 普通的数目是实数, 那么在实数域上, $x^2+1=0$ 就没有解. 在复数域上, 我们不但使它有解, 并且复数有非常巧妙的性质, 有很多现象都被放在复数里头了. 复数与实数一样, 有运算的规律, 你用这个规律之后, 复数代表了很多现象. 我们以后会看到在复数里头的这些内容. 所以, 数学要应用, 我们这个课是应用数学, 要学会应用. 要应用的话, 会发现复数很要紧. 因此, 复变函数论在 19 世纪的发展是数学里头最要紧的, 是一个比其它方面的发展来得更要紧一些的发展. 最后, 我得留点时间讲讲在复数方面的应用. 复数不只是使得对于任一个方程式有解, 并且利用复数, 很多数学问题来得简单. 复变函数论比实变函数论简单多了. 实变函数论有许多抽象的问题, 其实与实际不大有关系, 不过当时也需要了. 所以这是两个题目, 我要在这个课程里头把它们想办法讲一点, 使得大家能了解微积分在它们上的应用是最重要的两个方向.

复数的发现: 课外阅读 The Battle of the Cubic Equation and the Dawn of the Complex Numbers

几百年前人们在类似下面的方程的求解中引入了 $^3 x^2 + 1 = 0$ 虚数单位 (imaginary unit) i, 满足

$$i^2 = -1$$

于是可以定义复数 (complex numbers)

$$z = 1 \times (Rez) + i \times (Imz)$$

Rez 和 Imz 分别是 z 的实部和虚部。 课后阅读⁴

with the real valued factors respectively. By this definition, a complex number z can be decomposed into real numbers x, y, r and φ such that

In what follows, a very brief review of *complex analysis*, or, by another term, *function theory*, will be presented. For much more detailed introductions to complex analysis, including proofs, take, for instance, the "classical" books among a zillion of other very good ones. We shall study complex analysis not only for its beauty but also because it yields very important analytical methods and tools; for instance for the solution of (differential) equations and the computation of definite integrals. These methods will then be required for the computation of distributions and Green's functions, as well for the solution of differential equations of mathematical physics – such as the Schrödinger equation.

One motivation for introducing *imaginary numbers* is the (if you perceive it that way) "malady" that not every polynomial such as $P(x) = x^2 + 1$ has a root x – and thus not every (polynomial) equation $P(x) = x^2 + 1 = 0$ has a solution x – which is a real number. Indeed, you need the imaginary unit $i^2 = -1$ for a factorization P(x) = (x+i)(x-i) yielding the two roots $\pm i$ to achieve this. In that way, the introduction of imaginary numbers is a further step towards omni-solvability. No wonder that the fundamental theorem of algebra, stating that every non-constant polynomial with complex coefficients has at least one complex root – and thus total factorizability of polynomials into linear factors follows!

If not mentioned otherwise, it is assumed that the *Riemann surface*, representing a "deformed version" of the complex plane for functional purposes, is simply connected. Simple connectedness means that the Riemann surface is path-connected so that every path between two points can be continuously transformed, staying within the domain, into any other path while preserving the two endpoints between the paths. In particular, suppose that there are no "holes" in the Riemann surface; it is not "punctured."

Furthermore,

Note that the function $\varphi \mapsto e^{i\varphi}$ is not injective. In particular, $\exp(i\varphi) = \exp(i(\varphi + 2\pi k))$ for arbitrary $k \in \mathbb{Z}$. This has no immediate consequence on z; but it yields differences for *functions* thereof, like the square root or the logarithm. A remedy is the introduction of Riemann surfaces which are "extended" and "deformed" versions of the complex plane.

Many rules of classical arithmetic can be carried over to complex arithmetic. Note, however, that, because of noninjectivity of $\exp(i\varphi)$ for arbitrary values of φ , for instance, $\sqrt{a}\sqrt{b} = \sqrt{ab}$ is only valid if at least one factor a or b is positive; otherwise one could construct wrong deductions $-1 = i^2 \stackrel{?}{=} \sqrt{i^2}\sqrt{i^2} \stackrel{?}{=} \sqrt{-1}\sqrt{-1} \stackrel{?}{=} \sqrt{(-1)^2} = 1$. More generally, for two arbitrary numbers, u and v, $\sqrt{u}\sqrt{v}$ is not always equal to \sqrt{uv} . The

³虚数的引入还有其实在的意义,具体可以阅读一些关于三次方程的求解的历史。

⁴By combining imaginary and real numbers, any *complex number* can be defined to be some linear combination of the real unit number "1" with the imaginary unit number i that is,

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magical complex numbers

The magic number 'i',

How is it that -1 can have a square root? The square of a positive number is always positive, and the square of a negative number is again positive (and the square of 0 is just 0 again, so that is hardly of use to us here). It seems impossible that we can find a number whose square is actually negative. Yet, this is the kind of situation that we have seen before, when we ascertained that 2 has no square root within the system of rational numbers. In that case we resolved the situation by extending our system of numbers from the rationals to a larger system, and we settled on the system of reals. Perhaps the same trick will work again. Indeed it will. In fact what we have to do is something much easier and far less drastic than the passage from the rationals to the reals. (Raphael Bombelli introduced the procedure in 1572 in his work L'Algebra, following Gerolamo Cardano's original encounters with complex numbers in his Ars Magna of 1545.) All we need do is introduce a single quantity, called 'i', which is to square to -1, and adjoin it to the system of reals, allowing combinations of i with real numbers to form expressions such as

$$a + ib$$

where a and b are arbitrary real numbers. Any such combination is called a complex number. It is easy to see how to add complex numbers:

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

which is of the same form as before (with the real numbers a + c and b + d taking the place of the a and b that we had in our original expression). What about multiplication? This is almost as easy. Let us find the product of a + ib with c + id. We first simply multiply these factors, expanding the expression using the ordinary rules of algebra: ¹

$$(a+ib)(c+id) = ac + ibc + aid + ibid$$
$$= ac + i(bc + ad) + i^2bd$$

n'th root of a complex number z parameterized by many (indeed, an infinity of) angles φ is no unique function any longer, as $\sqrt[n]{z} = \sqrt[n]{|z|} \exp{(i\varphi/n + 2\pi i k/n)}$ with $k \in \mathbb{Z}$. Thus, in particular, for the square root with n = 2, $\sqrt{u}\sqrt{v} = \sqrt{|u||v|} \exp{[(i/2)(\varphi_u + \varphi_v)]} \exp{[i\pi(k_u + k_v)]}$. Therefore, with $u = -1 = \exp[i\pi(1 + 2k)]$ and $v = -1 = \exp[i\pi(1 + 2k')]$ and $v = -1 = \exp[i\pi(1 + 2k')]$

even and odd k + k', respectively.

For many mathematicians Euler's identity

$$e^{i\pi} = -1$$
, or $e^{i\pi} + 1 = 0$, (2.18)

is the "most beautiful" theorem

But $i^2 = -1$, so we can rewrite this as

$$(a+ib)(c+id) = (ac-bd) + i(bc+ad).$$

which is again of the same form as our original a + ib, but with ac - bd taking the place of a and bc + ad taking the place of b.

It is easy enough to subtract two complex numbers, but what about division? Recall that in the ordinary arithmetic we are allowed to divide by any real number that is not zero. Now let us try to divide the complex number a + ib by the complex number c + id. We must take the latter to be non-zero, which means that the real numbers c and d cannot both be zero. Hence $c^2 + d^2 > 0$, and therefore $c^2 + d^2 \neq 0$, so we are allowed to divide by $c^2 + d^2$. It is a direct exercise [4.1] to check (multiplying both sides of the expression below by c + id) that

$$\frac{(a+ib)}{(c+id)} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}.$$

This is of the same general form as before, so it is again a complex number.

When we get used to playing with these complex numbers, we cease to think of a + ib as a pair of things, namely the two real numbers a and b, but we think of a + ib as an entire thing on its own, and we could use a single letter, say z, to denote the whole complex number z = a + ib. It may